

Lie group analysis of a generalized Krichever-Novikov differential-difference equation

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Abstract

The symmetry algebra of the differential–difference equation

$$\dot{u}_n = [P(u_n)u_{n+1}u_{n-1} + Q(u_n)(u_{n+1} + u_{n-1}) + R(u_n)]/(u_{n+1} - u_{n-1}),$$

where P , Q and R are arbitrary analytic functions is shown to have the dimension $1 \leq \dim L \leq 5$. When P , Q and R are specific second order polynomials in u_n (depending on 6 constants) this is the integrable discretization of the Krichever–Novikov equation. We find 3 cases when the arbitrary functions are not polynomials and the symmetry algebra satisfies $\dim L = 2$. These cases are shown not to be integrable. The symmetry algebras are used to reduce the equations to purely difference ones. The symmetry group is also used to impose periodicity $u_{n+N} = u_n$ and thus to reduce the differential–difference equation to a system of N coupled ordinary three points difference equations.

1 Introduction

This article is part of a general program the aim of which is to apply continuous groups to study discrete phenomena. More specifically, the purpose is to use Lie groups to classify, simplify and ultimately solve difference, or differential-difference equations. For recent reviews of this program see [8, 9, 24, 28, 31, 32, 34] and for a broader review of the subject of symmetries and integrability of difference equation see the Lecture Notes [22].

One of the applications of Lie groups to differential equations is to perform a symmetry classification. Thus a differential equation, or a family of equations is considered, depending on some unspecified functions $\phi_a(\vec{x}_i, \vec{u}_i, \vec{u}_{\vec{x}}, \dots)$ where $\vec{x} \in \mathbb{R}^p$, $\vec{u} \in \mathbb{R}^q$ are the independent and dependent variables, respectively. The problem is to determine the Lie point symmetry group of the system for ϕ_a generic and then to find all sets of functions ϕ_a for which the symmetry group is larger. For examples of such studies treating partial differential equations see e.g. [1–5, 7, 10, 11, 13, 14, 16–18, 29]

Such a symmetry classification has been performed for certain discrete dynamical systems (molecular chains) in earlier publications [12, 21, 23, 36]. Since these articles were published the theory of symmetries of differential-difference equations has undergone further developments [25, 32] and the formalism has been simplified.

The purpose of this article is to perform a symmetry analysis of a specific differential-difference equation which we shall call the generalized Krichever-Novikov differential-difference equation

$$u_{n,t} \equiv \dot{u}_n = \frac{P(u_n)u_{n+1}u_{n-1} + Q(u_n)(u_{n+1} + u_{n-1}) + R(u_n)}{u_{n+1} - u_{n-1}}. \quad (1.1) \quad \{\text{EYdKN}\}$$

In (1.1) the functions $P(u_n)$, $Q(u_n)$ and $R(u_n)$ are arbitrary analytical functions. This equation is of physical and mathematical interest mainly because it generalizes the "Yamilov discretization of the Krichever–Novikov equation" [26, 33, 34]. In this case P , Q and R are restricted to

$$P_n = \alpha u_n^2 + 2\beta u_n + \gamma, \quad Q_n = \beta u_n^2 + \lambda u_n + \delta, \quad R_n = \gamma u_n^2 + 2\delta u_n + \omega, \quad (1.2)$$

and equation (1.1) is integrable and allows an infinite number of generalized symmetries. The Lie point symmetries of (1.1) when P , Q and R are arbitrary second order polynomials were studied in [26]. The dimension of the symmetry algebra L was found to satisfy $2 \leq \dim L \leq 5$. All cases with $\dim L = 5$ or 4 are integrable. Among the three cases with $\dim L = 3$ only two were integrable, among the nine equations with $\dim L = 2$ exactly three are integrable. In general, integrable equations (1.2) tend to have larger Lie point symmetries than non integrable ones. As opposed to non integrable ones, the integrable ones have infinite dimensional algebras of generalized symmetries.

Here we will concentrate on the case when P , Q and R are not of the form (1.2) or its generalization considered in [26] with generic second order polynomials and then compare the results with a symmetry analysis of the "generalized Krichever–Novikov" equation

$$v_t = v_{xxx} - \frac{3}{2} \frac{v_{xx}^2}{v_x} + \frac{f(v)}{v_x}, \quad (1.3) \quad \{\text{eq-gKN}\}$$

where $f(v)$ is an arbitrary analytical function. A symmetry analysis of (1.3) was performed in [6] and [26]. Equation (1.3) is obtained as a continuous limit of (1.1) as described in [26]. Thus, we put

$$\begin{aligned} u_n(t) &= v(x, t) \equiv v, & x &= nh + 6\frac{t}{h^2}, & k &= -\frac{12}{h^3} \\ P(u_n) &= k + 2hF(v), & Q(u_n) &= -kv + 2hG(v), & R(u_n) &= kv^2 + 2hH(v) \end{aligned} \quad (1.4)$$

and take the limit $h \rightarrow 0$, $n \rightarrow \infty$, hn finite. In the limit we obtain (1.3) with

$$f(v) = v^2 P(v) + 2vQ(v) + R(v). \quad (1.5) \quad \{\mathbf{fu}\}$$

The original (integrable) Krichever–Novikov equation [20] has the form (1.3) where f is a fourth degree polynomial with arbitrary constant coefficients. It firstly appeared in a study of quasi periodic solutions of soliton equations [19, 20]. For further references see [35] and also [26].

In Section 2 we derive the determining equations for the symmetries of (1.1) and the "allowed transformations" that leave the form of (1.1) invariant for arbitrary functions P , Q and R . The determining equations are solved in Section 3. Eq. (1.1) is always invariant under time translations and integer shifts in n . We find all functions P , Q and R for which the symmetry algebra is larger than $X_0 = \partial_t$. We present only the cases when P , Q and R are not all second order polynomials since such polynomial cases were already treated in [26]. The 5 non polynomial cases are summed up in Section 4. The symmetry algebra in all non polynomial cases is two-dimensional. Among the 5 cases two are redundant in that they are related by allowed transformations. The 3 representative non polynomial cases are presented in Table 2. In Section 5 we analyze the results and show that none of the non polynomial cases is integrable. We show how the symmetry group can be used to reduce (1.1) to a one variable difference equation. Invariance under shifts of the discrete variable n can be used to impose periodicity, i.e. reduce (1.1) to a finite system of ordinary differential equations. Some conclusions are presented in the final Section 6.

2 Determining equations and allowed transformations

Equation (1.1) belongs to a class of equations studied in [25] where it was shown that the vector fields generating their symmetries will have the general form

$$X = \tau(t)\partial_t + \phi_n(t, u_n)\partial_{u_n} \quad (2.1) \quad \{\mathbf{fieldX}\}$$

i.e. $\tau(t)$ does not depend on n or u_n . The prolongation of X acting on (1.1) is

$$\begin{aligned} prX &= \tau(t)\partial_t + \sum_{j=n-1}^{n+1} \phi_j(t, u_j)\partial_{u_j} + \phi_n^{(1)}\partial_{\dot{u}_n} \\ \phi_n^{(1)} &= D_t\phi_n(t, u_n) - [D_t\tau(t)]\dot{u}_n \end{aligned} \quad (2.2)$$

where D_t is the total differentiation operator. We require that prX should annihilate (1.1) on its solution set. This implies the determining equation

$$\begin{aligned} \phi_{n,t}(u_{n+1} - u_{n-1})^2 &+ (\phi_{n,u_n} - \dot{\tau}) [Pu_{n+1}u_{n-1} + Q(u_{n+1} + u_{n-1}) + R] (u_{n+1} - u_{n-1}) \\ &- \phi_n [P_{,u_n}u_{n+1}u_{n-1} + Q_{,u_n}(u_{n+1} + u_{n-1}) + R_{,u_n}] (u_{n+1} - u_{n-1}) \\ &= \phi_{n-1} [Pu_{n+1}^2 + 2Qu_{n+1} + R] - \phi_{n+1} [Pu_{n-1}^2 + 2Qu_{n-1} + R]. \end{aligned} \quad (2.3)$$

Equation (1.1) is form invariant under the group of Möbius transformations of u_n and linear transformations of time

$$u_n = \frac{\xi_1 u_n^* + \xi_2}{\xi_3 u_n^* + \xi_4}, \quad t = \theta_1 t^* + \theta_2, \quad (2.4)$$

$$\xi_1 \xi_4 - \xi_2 \xi_3 = \pm 1, \quad \theta_1 \neq 0, \quad \xi_i, \theta_i \in \mathbb{R}.$$

We shall call (2.4) "allowed transformations".

Eq. (1.1) is transformed into

$$u_{n,t^*}^* = \frac{\tilde{P}(u_n^*)u_{n+1}^*u_{n-1}^* + \tilde{Q}(u_n^*)(u_{n+1}^* + u_{n-1}^*) + \tilde{R}(u_n^*)}{u_{n+1}^* - u_{n-1}^*} \quad (2.5) \quad \{\mathbf{U*nt}\}$$

with

$$\begin{aligned} \tilde{P}(u_n^*) &= (\xi_3 u_n^* + \xi_4)^2 [P^*(u_n)\xi_1^2 + 2Q^*(u_n)\xi_1\xi_3 + R^*(u_n)\xi_3^2] \\ \tilde{Q}(u_n^*) &= (\xi_3 u_n^* + \xi_4)^2 [P^*(u_n)\xi_1\xi_2 + Q^*(u_n)(\xi_1\xi_4 + \xi_2\xi_3) + R^*(u_n)\xi_3\xi_4] \\ \tilde{R}(u_n^*) &= (\xi_3 u_n^* + \xi_4)^2 [P^*(u_n)\xi_2^2 + 2Q^*(u_n)\xi_2\xi_4 + R^*(u_n)\xi_4^2]. \end{aligned} \quad (2.6)$$

In (2.6) $P(u_n)$, $Q(u_n)$ and $R(u_n)$ are to be viewed as the same functions as in (1.1), however they depend on u_n^* via the Möbius transformation (2.4) e.g we have $P^*(u_n) \equiv P(\frac{\xi_1 u_n^* + \xi_2}{\xi_3 u_n^* + \xi_4})$.

In particular for a pure inversion we have $\xi_2 = \xi_3 = 1, \xi_1 = \xi_4 = 0, \theta_1 = 1$ and $\theta_2 = 0$:

$$\tilde{P}(u_n^*) = (u_n^*)^2 R(\frac{1}{u_n^*}), \quad \tilde{Q}(u_n^*) = (u_n^*)^2 Q(\frac{1}{u_n^*}), \quad \tilde{R}(u_n^*) = (u_n^*)^2 P(\frac{1}{u_n^*})$$

From (2.6) we see that with no loss of generality we can assume

$$P(u_n) \neq 0 \quad (2.7) \quad \{\mathbf{Pnonzero}\}$$

in (1.1) (and in (2.3)). Indeed if we have $P(u_n) = 0$ in (1.1) then the Möbius transformation will generate $\tilde{P}(u_n^*) \neq 0$ (unless all three functions satisfy $P(u_n) = Q(u_n) = R(u_n) = 0$).

The quantities u_{n+1} and u_{n-1} appear in (2.3) explicitly. Implicitly they figure only via ϕ_{n+1} and ϕ_{n-1} , respectively. Taking the derivative $\partial_{u_{n+1}}^2 \partial_{u_{n-1}}^2$ we obtain (for $P(u_n) \neq 0$)

$$\phi_{n+1, u_{n+1} u_{n+1}} - \phi_{n-1, u_{n-1} u_{n-1}} = 0 \quad (2.8) \quad \{\mathbf{eq2-9}\}$$

From (2.8) and (2.3) we obtain the following result

Theorem 2.1. *The Lie algebra of local Lie point symmetries of the Generalized Krichever–Novikov Differential–Difference equation (1.1) is realized by vector fields f the form (2.1) with*

$$\begin{aligned} \tau &= \tau_1 t + \tau_0, & \phi_n &= \alpha_n + \beta_n u_n + \gamma_n u_n^2 \\ \alpha_n &= a_1 + (-1)^n a_2, & \beta_n &= b_1 + (-1)^n b_2, & \gamma_n &= c_1 + (-1)^n c_2 \end{aligned} \quad (2.9)$$

where $\tau_1, \tau_0, a_1, a_2, b_1, b_2$, and c_1, c_2 are constants.

This theorem was already presented in Ref [26] for P, Q and R second order polynomials. Here we have generalized it to arbitrary analytic functions of u_n .

To proceed further we substitute expressions (2.9) into the determining equations (2.3) and collect independent powers of the form $u_{n+1}^p u_{n-1}^q$ for $0 \leq p \leq 2, 0 \leq q \leq 2$. Terms $u_{n+1}^2 u_{n-1}^2$ cancel, the remaining terms contain an overall factor $(u_{n+1} - u_{n-1})$ that can be dropped. We are left with three equations, the coefficients of $u_{n+1} u_{n-1}$, $u_{n+1} + u_{n-1}$ and $(u_{n+1})^0 (u_{n-1})^0$, respectively. Since the functions $P(u_n), Q(u_n), R(u_n)$ are assumed to be analytical and n -independent, the only dependence on $(-1)^n$ in the determining equations is explicit (i.e. not contained in P, Q and R). Hence the terms with and without the factor $(-1)^n$ must vanish separately. Finally we obtain six determining equations:

$$-a_1 P + c_1 R - (a_1 + b_1 u_n + c_1 u_n^2) Q' + (b_1 + 2u_n c_1 - \tau_1) Q = 0 \quad (2.10)$$

$$a_2 P - c_2 R - (a_2 + b_2 u_n + c_2 u_n^2) Q' + (b_2 + 2c_2 u_n) Q = 0 \quad (2.11)$$

$$2c_1 Q - (a_1 + b_1 u_n + c_1 u_n^2) P' + (2c_1 u_n - \tau_1) P = 0 \quad (2.12)$$

$$-2c_2 Q - (a_2 + b_2 u_n + c_2 u_n^2) P' + (2b_2 + 2c_2 u_n) P = 0 \quad (2.13)$$

$$-2a_1 Q - (a_1 + b_1 u_n + c_1 u_n^2) R' + (2b_1 + 2c_1 u_n - \tau_1) R = 0 \quad (2.14)$$

$$2a_2 Q - (a_2 + b_2 u_n + c_2 u_n^2) R' + 2c_2 u_n R = 0 \quad (2.15)$$

We see that the constant τ_0 does not figure in (2.10 – 2.15) whereas all other constants determining the symmetry vector field

$$X = (\tau_1 t + \tau_0) \partial_t + [(a_1 + b_1 u_n + c_1 u_n^2) + (-1)^n (a_2 + b_2 u_n + c_2 u_n^2)] \partial_{u_n} \quad (2.16)$$

are present. Hence the vector field

$$T_0 = \partial_t \quad (2.17)$$

corresponding to $\tau_0 = 1$ is always present in the symmetry algebra for arbitrary functions P, Q and R . The existence of further symmetries imposes conditions on these functions.

Our strategy will be to consider (2.10–2.15) as a system of coupled ordinary differential equations for functions P, Q and R . We solve them for these functions, treating the constants τ_1, a_i, b_i, c_i as parameters. Then we use the Möbius transformations and time dilations to simplify the obtained expressions and re-parametrize them. Finally, we reintroduce the obtained functions into (2.10–2.15) and solve these equations for the parameters τ_1, a_i, b_i and c_i and thus determine the symmetry algebra in each case. We are only interested in cases when at least one of the functions P, Q and R is not a second order polynomial. Once the functions P, Q and R and the corresponding symmetry algebras of eq. (1.1) are obtained we check for redundancies. If different equations (with isomorphic symmetry algebras) can be transformed into each other by an allowed transformation, we keep only one of them in the final representative list.

3 Solution of determining equation

In order to solve the determining equations we split the process into several sub-cases. We are interested only in real solutions ($P, Q, R \in \mathbb{R}$).

3.1 Generic case $c_1 \neq 0$ and $c_2 \neq 0$

We multiply (2.12) by c_2 , (2.13) by c_1 and add the two equations and obtain a first order ODE (depending on 2 parameters κ and ρ)

$$(u_n^2 + \kappa)P'(u_n) - (2u_n + \rho)P(u_n) = 0, \quad \kappa = 1, -1, 0, \quad \rho \in \mathbb{R} \quad (3.1) \quad \{\mathbf{C1-e1}\}$$

The solutions are

$$\begin{aligned} \kappa &= 1 & P(u_n) &= (u_n^2 + 1)e^{\rho \arctan u_n} \\ \kappa &= -1 & P(u_n) &= (u_n - 1)^{1+\frac{\rho}{2}}(u_n + 1)^{1-\frac{\rho}{2}} \\ \kappa &= 0 & P(u_n) &= u_n^2 e^{-\frac{\rho}{u_n}} \end{aligned} \quad (3.2)$$

We will consider $\rho \neq 0$ first and each value of κ separately. Then we consider $\rho = 0$ and each κ again.

i) $\rho \neq 0, \kappa = 1$

$$\begin{aligned} P(u_n) &= (u_n^2 + 1)e^{\rho \arctan u_n} \\ Q(u_n) &= \left(-\frac{1}{2}\rho u_n^2 + Au + B\right)e^{\rho \arctan u_n} \\ R(u_n) &= \frac{1}{2(4 + \rho^2)(u_n^2 + 1)} ((\rho^4 + 6\rho^2 + 8)u_n^4 + Cu_n^3 + Du_n^2 + Eu_n + F) e^{\rho \arctan u_n}. \end{aligned} \quad (3.3)$$

Requiring that all equations (2.10–2.15) are satisfied implies $c_1 = 0$, $\rho = 0$ or ρ complex number, i.e. disagree with the assumptions.

ii) $\rho \neq 0, \kappa = -1$

$$\begin{aligned} P(u_n) &= (u_n - 1)^{1+\frac{\rho}{2}}(u_n + 1)^{1-\frac{\rho}{2}} \\ Q(u_n) &= \frac{(u_n - 1)^{1+\frac{\rho}{2}}(u_n + 1)^{1-\frac{\rho}{2}}}{u_n^2 - 1} \left(-\frac{1}{2}\rho u_n^2 + Au + B\right) \\ R(u_n) &= \frac{(u_n - 1)^{1+\frac{\rho}{2}}(u_n + 1)^{1-\frac{\rho}{2}}}{2(u_n^2 - 1)^2(\rho^2 - 4)} ((\rho^4 + 6\rho^2 + 8)u_n^4 + Cu_n^3 + Du_n^2 + Eu_n + F) \end{aligned} \quad (3.4)$$

We find again that if we require for all equations (2.10–2.15) to be satisfied, there is no solution satisfying all assumptions.

iii) $\rho \neq 0, \kappa = 0$

$$P(u_n) = u_n^2 e^{-\frac{\rho}{u_n}} \quad (3.5)$$

$$Q(u_n) = \left(-\frac{1}{2}\rho u_n^2 + Au + B \right) e^{-\frac{\rho}{u_n}} \quad (3.6)$$

$$R(u_n) = \frac{1}{u_n^2 \rho (\rho^2 - 4)} (Cu_n^4 + Du_n^3 + Eu_n^2 + Fu_n + G) e^{-\frac{\rho}{u_n}} \quad (3.7)$$

Again equations (2.10–2.15) cannot be satisfied. To conclude: there is no function $P(u_n) \neq 0$ for which we have $c_1 c_2 \rho \neq 0$.

Considering $\rho = 0$ we obtain three additional cases (for $\kappa = \pm 1, 0$). However, in all cases (2.10–2.15) imply that $P(u_n), Q(u_n)$ and $R(u_n)$ are second order polynomials, a case already fully analyzed in [26].

3.2 Case $c_1 \neq 0$ and $c_2 = 0$

Equation (2.13) will be reduced to

$$(b_2 u_n + a_2) P'(u_n) - 2b_2 P(u_n) = 0 \quad (3.8) \quad \{\text{eP-case2}\}$$

i) $b_2 \neq 0$.

This case again leads to P, Q and R that are (at most) second order polynomials.

ii) $b_2 = 0, a_2 \neq 0$

The only solution is $P(u_n) = Q(u_n) = R(u_n) = 0$.

iii) $a_2 = 0, b_2 = 0$

We have 3 equations left (2.10, 2.12, 2.14). Using allowed transformations we can shift $b_1 \rightarrow 0$ and $c_1 \rightarrow 1$. The solution of the three remaining equations will depend on the value of a_1 . We need to consider three separate cases $a_1 = \pm 1, 0$

• $a_1 = 1$

We have

$$\begin{aligned} P(u_n) &= A e^{\rho \arctan u} (1 + u_n^2), \quad \rho \neq 0, \\ Q(u_n) &= 0 \\ R(u_n) &= A e^{\rho \arctan u} (1 + u_n^2) \end{aligned} \quad (3.9)$$

We get that $\tau_1 = -\rho$ and $a_1 = 1, b_1 = 0, c_1 = 1$ and we already have $a_2 = b_2 = c_2 = 0$.

• $a_1 = -1$

We have

$$\begin{aligned}
P(u_n) &= \left(\frac{1+u_n}{1-u_n}\right)^\rho (A + Bu_n + Cu_n^2), \quad \rho \neq 0 \\
Q(u_n) &= -\frac{1}{2} \left(\frac{1+u_n}{1-u_n}\right)^\rho (Bu_n^2 + (2A + 2C)u_n + B) \\
R(u_n) &= \left(\frac{1+u_n}{1-u_n}\right)^\rho (C + Bu_n + Au_n^2)
\end{aligned} \tag{3.10}$$

We also get $\tau_1 = 2\rho$ and $a_1 = -1, b_1 = 0, c_1 = 1$ and we assumed $a_2 = b_2 = c_2 = 0$.

For special case $\rho = 1$ we obtain again polynomial solutions for P, Q and R of (at most) 2nd degree.

$$\bullet a_1 = 0$$

$$\begin{aligned}
P(u_n) &= e^{\frac{\rho}{u_n}} (A + Bu_n + Cu_n^2), \quad \rho \neq 0 \\
Q(u_n) &= -\frac{1}{2} e^{\frac{\rho}{u_n}} (Bu_n^2 + 2Au_n) \\
R(u_n) &= e^{\frac{\rho}{u_n}} Au_n^2
\end{aligned} \tag{3.11}$$

We obtain $\tau_1 = \rho$ and $a_1 = 0, b_1 = 0, c_1 = 1$ and we assumed $a_2 = b_2 = c_2 = 0$. The coefficient ρ can be rescaled to $\rho = 1$ by an allowed transformation $u_n \rightarrow \rho u_n$.

3.3 Case $c_1 = 0$ and $c_2 \neq 0$

Equation (2.12) reduces to

$$(b_1 u_n + a_1)P'(u_n) + \tau_1 P(u_n) = 0 \tag{3.12} \quad \{\text{c1-0c2-no}\}$$

and we must substitute its solutions into the system (2.10–2.15). Running through all possible cases we find:

- i) Unless (3.12) is solved trivially P, Q and R are second order polynomials.
- ii) Eq. (3.12) is solved trivially leaving $P(u_n)$ arbitrary if $b_1 = a_1 = \tau_1 = 0$. We are left with only three equations (2.11, 2.13, 2.15). Their solution is again a set of second order polynomials. Thus, no new solutions are obtained for $c_2 \neq 0$.

3.4 Case $c_1 = 0$ and $c_2 = 0$

The equations (2.10–2.15) simplify to:

$$-a_1P - (a_1 + b_1u_n)Q' + (b_1 - \tau_1)Q = 0 \quad (3.13)$$

$$a_2P - (a_2 + b_2u_n)Q' + b_2Q = 0 \quad (3.14)$$

$$-(a_1 + b_1u_n)P' - \tau_1P = 0 \quad (3.15)$$

$$-(a_2 + b_2u_n)P' + 2b_2P = 0 \quad (3.16)$$

$$-2a_1Q - (a_1 + b_1u_n)R' + (2b_1 - \tau_1)R = 0 \quad (3.17)$$

$$2a_2Q - (a_2 + b_2u_n)R' = 0 \quad (3.18)$$

There are several cases to consider, but only two of them give new results, namely

- i) $b_2 = 0, a_2 = 0, b_1 \neq 0$

From (3.15) we get

$$\begin{aligned} P(u_n) &= Au_n^\rho, \quad \rho \neq 0, \\ Q(u_n) &= Bu_n^{1+\rho} \\ R(u_n) &= Cu_n^{2+\rho}. \end{aligned} \quad (3.19)$$

We find that $a_1 = 0, \tau_1 = -\rho b_1$, with b_1 free, and $b_1 \neq 0$.

- ii) $b_1 = 0, b_2 = 0, a_2 = 0$

We consider $a_1 \neq 0$ as $a_1 = 0$ gives us equation $\tau_1P = 0, \tau_1R = 0, \tau_1Q = 0$. For $a_1 \neq 0$ the three remaining equations imply

$$\begin{aligned} P(u_n) &= Ae^{\rho u_n}, \quad \rho \neq 0, \\ Q(u_n) &= e^{\rho u_n}(-Au_n + B) \\ R(u_n) &= e^{\rho u_n}(Au_n^2 - 2Bu_n + C) \end{aligned} \quad (3.20)$$

After substituting back we obtain $\tau_1 = -a_1\rho$, a_1 is free and we also had $c_1 = c_2 = b_1 = b_2 = a_2 = 0$. We can put $\rho = 1, A = 1$ and $C = 0$ by an allowed transformation of the form $u_n \rightarrow u_n + \beta, t \rightarrow \gamma t$.

4 Results of the symmetry classification

Let us now sum up and analyze the results of Section 3.

1. We have verified the entire symmetry classification of [26], where P, Q and R are second order polynomials. A brief summary is given in Table 1.

dim L	Number of cases	Equation in Ref. [26]
5	1	(4.1)
4	2	(4.2), (4.6)
3	3	(4.3), (4.4), (4.5)
2	9	(4.8) (7 cases), (4.10), (4.11)

Table 1: Summary of cases with P , Q and R polynomials (of second order) and symmetry algebra L satisfying $\dim L \geq 2$.

2. We have obtained 5 cases when $P(u_n)$ is not a second order polynomial and the symmetry algebra satisfies $\dim L \geq 2$. They correspond to (3.9–3.11, 3.19) and (3.20), respectively. All of them have two dimensional non-Abelian symmetry algebras of the form

$$X_0 = \partial_t, \quad X_1 = t\partial_t + (Au_n^2 + Bu_n + C)\partial_{u_n}, \quad (4.1)$$

where A , B and C are some specific constants. The vector fields $(u_n^2\partial_{u_n}, u_n\partial_{u_n}, \partial_{u_n})$ span an $sl(2, \mathbb{R})$ algebra. The Möbius transformations (2.4) represents the Lie group $SL(2, \mathbb{R})$ that acts on these vector fields by inner automorphisms. Hence we can use this Möbius transformation to transform the term $(Au_n^2 + Bu_n + C)\partial_{u_n}$ into a representative of one of the three distinct one-dimensional subalgebras of $sl(2, \mathbb{R})$, namely: $(1 + u_n^2)\partial_{u_n}$ (rotation $o(2)$), $u_n\partial_{u_n}$ (pseudorotations $o(1, 1)$), or ∂_{u_n} (unipotent transformations). This suggests that some of the five obtained generalized Krichever–Novikov equations may be redundant, i.e. related to each other by Möbius transformations and this is indeed the case.

The compact subalgebra $o(2)$ occurs just once, namely in the case (3.9). The functions P , Q and R depend on one constant A . By a time dilation we can set $A = 1$ and have

$$X_1 = t\partial_t - \frac{1}{\rho}(1 + u_n^2)\partial_{u_n}. \quad (4.2)$$

The non compact subalgebra $o(1, 1)$ occurs twice. First in (3.10) with

$$X_1 = t\partial_t + \frac{1}{2\rho}(-1 + u_n^2)\partial_{u_n}, \quad (4.3)$$

and secondly in (3.19) with

$$X_1 = t\partial_t - \frac{1}{\rho}u_n\partial_{u_n}. \quad (4.4)$$

The Möbius transformation $u_n \rightarrow \frac{1-u_n}{1+u_n}$ takes the vector field (4.3) and also the functions P , Q and R in (3.10) into those in (3.19) (with a redefinition of the constants).

N_0	The equation	Basis elements	Continuous limit
1	(5.1)	$X_1 = t\partial_t - \frac{1}{\rho}(1 + u_n^2)\partial_{u_n}$	$(v^2 + 1)^2 e^{p \arctan v}$
2	(5.2)	$X_1 = t\partial_t - \frac{1}{\rho}u_n\partial_{u_n}$	v^p
3	(5.3)	$X_1 = t\partial_t - \partial_{u_n}$	e^v

Table 2: Representative list of equations with non polynomial functions P , Q and R and $\dim L > 1$ (in all cases we have $\dim L = 2$). The last column shows the function $f(v)$ in the continuous limit (1.3).

The nilpotent subalgebra also occurs twice, in (3.11) with

$$X_1 = t\partial_t + u_n^2\partial_{u_n}, \quad (4.5)$$

and in (3.20) with

$$X_1 = t\partial_t - \partial_{u_n}. \quad (4.6)$$

The Möbius transformation $u_n \rightarrow \frac{1}{u_n}$ takes the vector field (4.6) into (4.5) and also the functions P , Q and R of (3.11) into those of (3.20).

Finally we are left with three inequivalent cases, summed up in Table 2. In each case we can calculate the continuous limit as indicated in (1.4, 1.5) of the Introduction. The results are included in Table 2. Using a rescaling of time we can always rescale $A \rightarrow 1$.

5 Analysis of non polynomial cases

The generalized Krichever–Novikov differential–difference equation is always invariant under time translations (since the coefficients do not depend on t). The main result of this article is that we have identified all cases, when the symmetry group is larger, i.e. its Lie point symmetry algebra L contains at least one additional elements. In addition to known cases [26] summed up in Table 1 we have identified three new ones given in Table 2.

Let us now analyze the new differential–difference equations, namely

$$\dot{u}_n = e^{\rho \arctan u_n} \frac{(u_{n+1}u_{n-1} + 1)(1 + u_n^2)}{u_{n+1} - u_{n-1}}, \quad \rho \neq 0, \quad (5.1)$$

$$\dot{u}_n = u_n^\rho \frac{u_{n+1}u_{n-1} + Au_n(u_{n+1} + u_{n-1}) + Bu_n^2}{u_{n+1} - u_{n-1}}, \quad \rho \neq 0, \quad (5.2)$$

$$\dot{u}_n = e^{u_n} \frac{u_{n+1}u_{n-1} + (-u_n + A)(u_{n+1} + u_{n-1}) + u_n^2 - 2Au_n}{u_{n+1} - u_{n-1}}. \quad (5.3)$$

1. The first question concerns their integrability. In the original Yamilov discretization of the Krichever–Novikov equation [30, 33, 34] it was shown that for P , Q and R satisfying (1.2) equation (1.1) is integrable and allows an infinity of commuting generalized symmetries. Let us now consider (5.1), (5.2) and (5.3) from this point of view. From [27] we know that any equation of the Volterra type $\dot{u}_n = f(u_{n-1}, u_n, u_{n+1})$ which has higher generalized symmetries has to satisfy a set of conditions given in (3.10) of [27]. The first of them is

$$\frac{\delta}{\delta u_n} \partial_t \log \frac{\partial f}{\partial u_{n+1}} = 0, \quad (5.4)$$

where the variational derivative operator is given by $\frac{\delta}{\delta u_n} a_n(u_{n-i}, \dots, u_{n+j}) = \sum_{k=n-i}^{n+j} \frac{\partial a_k}{\partial u_n}$ with i, j positive numbers. Applying this condition to the different f 's corresponding to the three non polynomial cases (5.1 – 5.3) we find that this condition is never satisfied and consequently the three non polynomial cases are not integrable. This was to be expected as they are not contained in the complete classification of Volterra type equations up to point and Miura transformations performed by Yamilov [33] and reviewed in [34].

2. The symmetries of the obtained differential–difference equations can be used to perform symmetry reduction.

Eq. (5.1) is invariant under the transformations induced by the vector field $X = -\rho t \partial_t + (u_n^2 + 1) \partial_{u_n}$. Invariant solutions will have the form

$$u_n = \tan(\gamma_n - \frac{1}{\rho} \log t). \quad (5.5)$$

Substituting (5.5) into (5.1) we find that γ_n must satisfy the nonlinear recursion relation

$$\tan(\gamma_{n+1} - \gamma_{n-1}) = -\rho e^{\rho \gamma_n}. \quad (5.6)$$

From eq. (5.2) the vector field $X = -\rho t \partial_t + u_n \partial_{u_n}$ provides the reduction formula

$$u_n = \gamma_n t^{-\frac{1}{\rho}}, \quad (5.7)$$

where γ_n must satisfy the recursion relation

$$\frac{\gamma_n^{\rho-1}}{\gamma_{n+1} - \gamma_{n-1}} [\gamma_{n+1} \gamma_{n-1} + A \gamma_n (\gamma_{n+1} + \gamma_{n-1}) + B \gamma_n] = -\frac{1}{\rho}. \quad (5.8)$$

Finally, for eq. (5.3) the vector field $X = t \partial_t - \partial_{u_n}$ provide the reduction formula

$$u_n = \gamma_n - \log t, \quad (5.9)$$

and the following recursion relation for γ_n :

$$\frac{e^{\gamma_n}}{\gamma_{n+1} - \gamma_{n-1}} [\gamma_{n+1}\gamma_{n-1} + (A - \gamma_n)(\gamma_{n+1} + \gamma_{n-1}) + \gamma_n^2 - 2A\gamma_n] = -1. \quad (5.10)$$

Thus, in each case invariance under the subgroup corresponding to the vector field X_1 leads to the reduction of the differential–difference equation to a three term nonlinear difference equation.

It is interesting to compare the symmetry algebra of eq. (5.1), (5.2) and (5.3) with those of their continuous limits, given in Table 1 of Ref. [26]. The symmetry algebras $\{X_0, X_1\}$ of Table 2 of this article survive (and remain the same) in this limit. An additional vector field $X_2 = \partial_x$ appears in the limit. The corresponding group of transformation $n \rightarrow n + N$, $N \in \mathbb{Z}$ is also present in the discrete case (1.1) (for arbitrary functions P , Q and R). It is this symmetry that allows the construction of "periodic systems" satisfying

$$u_{n+N}(t) = u_n(t). \quad (5.11)$$

For $N = 1$ and $N = 2$ (1.1) implies $\dot{u}_n = 0$, and we obtain trivial difference equations.

For $N \geq 3$ we obtain a coupled system of N differential equations. For example for $N = 3$ the system is

$$\begin{aligned} \dot{u}_0 &= \frac{P(u_0)u_1u_2 + Q(u_0)(u_1 + u_2) + R(u_0)}{u_2 - u_1}, \\ \dot{u}_1 &= \frac{P(u_1)u_2u_0 + Q(u_1)(u_2 + u_0) + R(u_1)}{u_0 - u_2}, \\ \dot{u}_2 &= \frac{P(u_2)u_0u_1 + Q(u_2)(u_0 + u_1) + R(u_2)}{u_1 - u_0}. \end{aligned} \quad (5.12)$$

6 Conclusions

The main result of this paper are the three equations (5.1–5.3) and their symmetry algebras $\{X_0 = \partial_t, X_1\}$ with X_1 given in (4.2), (4.4) or (4.6), respectively. As shown in Section 5, these symmetry algebras can always be put to good use, just as in the case of differential equations.

Let us end with a rather general comment. Eq. (1.1), studied in this article, is a particular discretization (from two continuous variables to one continuous and one discrete) of the generalized Krichever–Novikov equation. The form of the discretization was inspired by Yamilov’s discretization [33] of the original integrable Krichever–Novikov equation [20]. Yamilov’s discretization preserved integrability, in particular an infinite Abelian algebra of generalized symmetries. Here we have shown that this discretization also preserves all point symmetries on a regular (equally spaced) lattice. This is to be contrasted with the general fact that a complete discretization of

an ODE or PDE preserving point symmetries requires the introduction of non uniform symmetry adapted lattices [8, 22, 24, 31].

An alternative point of view is that the PDE (1.3) is the continuous limit of the differential–difference equation (1.1) and that all point symmetries of the continuous limit are obtained from the point symmetries of (1.1). This is not the case for all differential–difference equations. Indeed it was shown in [15] that the Toda lattice has the potential Korteweg–de Vries equation as a continuous limit. Two of the five Lie point symmetries of this KdV equation are however obtained from generalized symmetries of the Toda lattice, not from point ones.

The relation between point symmetries of difference equations or differential–difference equations and their continuous limits needs a more general study.

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